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# Linear and quadratic invariants for the transformed Tavis–Cummings model

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**Abstract**

In the present communication we introduce the transformed Tavis–Cummings problem as a physical model to discuss constants of the motion (invariants) for such a system. The Hamiltonian we have used can be regarded as a most general time-dependent frequency converter model. The advantage of the present work is to handle a real physical problem which represents the interaction between two coupled oscillators. In this context we obtained real and complex classes of linear and quadratic invariants. We have employed the real quadratic invariants to define a new Dirac operator, from which the wavefunction in the coherent states is obtained.

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## 1. Introduction

The construction of invariant operators in quantum mechanics has attracted much attention in the last few decades. The simplicity of the rules for constructing these invariants and the instructive relation of the invariants to the asymptotic expansion of adiabatic invariant theory have stimulated an interest in using the invariants to solve some explicit quantum-mechanical problems [1, 2]. Exact invariants for time-dependent systems are decisive for investigating the physical properties of these systems. The simplest example illustrating this is the usual time-dependent harmonic oscillator [3–13]. The author of [1] introduced in his papers the role of these invariant operators, in which he describes a quantum system governed by a time-dependent Hamiltonian. The author showed that, if the system admits an invariant $\hat{I}(t)$ among its observables, it is possible to find a privileged basis of eigenstates of this operator, where the expansion of the state vector on this basis can be performed with independent coefficients. He in fact handled two different problems in his papers; the first was the time-dependent harmonic oscillator, while the second was the charged particle in a particular type of time-dependent classical electromagnetic field. On the other hand, one can see that the diagonalization of a quadratic Hamiltonian $\hat{H} = z^{\mathrm{T}} A z^{\mathrm{T}}$, $z = (\hat{q}_1, \ldots, \hat{q}_n, \hat{p}_1, \ldots, \hat{p}_n)$ with $A$ a real, symmetric and possibly time-dependent matrix, is important because it leads to a direct construction of all the possible invariants and constants of the motion. Alternative treatments of invariants are less systematic. Any time dependence in $\hat{H}$ can often be eliminated by canonical transformation. If this cannot be done, the eigenfrequencies (or growth rates if $\hat{H}$ is not positive definite) are time-dependent. Many authors have discussed the solution of the Schrödinger equation corresponding to $\hat{H}$, often from a group-theoretical point of view [14–16]. The reduction of such a Hamiltonian to diagonal form is achieved by means of canonical transformations, which constitute the symplectic group, and the problem may be reduced to a classification of the Lie algebra $sp(2n, \mathcal{R})$ into conjugacy classes [17]. As far as one can see, most of the previous work concentrated on one-dimensional time-dependent systems. However, in this context we follow up this work and try to find the constants of the motion (invariants) for different types of problem (two-dimensional time-dependent problem). The problem we consider represents one of the most fundamental problems in the field of quantum optics, that is the interaction between atoms and a field. This problem can be described by a Hamiltonian representing the generalized model of $N\,(>1)$ two-level atoms occupying the same site and collectively interacting with a quantized single mode of the field (in the absence of any radiation damping). This model is known as a generalized Tavis and Cummings (TC) model. The Hamiltonian representing such a system takes the form

$$\frac{\hat{H}}{\hbar} = \omega \hat{a}^{\dagger}\hat{a} + \frac{1}{2}\omega_0 \hat{J}_z + \lambda(\hat{a}^{\dagger m}\hat{J}_- + \hat{a}^m \hat{J}_+) \tag{1.1}$$

where $\hat{a}^{\dagger}$ and $\hat{a}$ are the boson creation and annihilation operators for the single mode of the field respectively. The $\hat{J}$ operators are the collective angular momentum operators, $\omega$ and $\omega_0$ are the field and the atomic frequencies, while $\lambda$ is the coupling constant. Note that, for $m = 1$, equation (1.1) reduces to the usual (TC) model. The operators $\hat{a}^{\dagger}$ and $\hat{a}$ satisfy the commutation relation $[\hat{a}, \hat{a}^{\dagger}] = 1$, while the operators $\hat{J}_-$, $\hat{J}_+$ and $\hat{J}_z$ satisfy the commutation relations $[\hat{J}_+, \hat{J}_-] = 2\hat{J}_z$ and $[\hat{J}_{\pm}, \hat{J}_z] = \mp\hat{J}_{\pm}$. The equations of motion in the Heisenberg picture for the Hamiltonian (1.1) are

$$\frac{\mathrm{d}\hat{a}}{\mathrm{d}t} = -\mathrm{i}\omega\hat{a} - \mathrm{i}\lambda m \hat{a}^{\dagger(m-1)}\hat{J}_- \qquad \frac{\mathrm{d}\hat{J}_-}{\mathrm{d}t} = -\mathrm{i}\omega_0 \hat{J}_- + \mathrm{i}\lambda \hat{a}^m \hat{J}_z. \tag{1.2}$$

The above equations are nonlinear ordinary differential equations and due to the nature of the nonlinearity it is not an easy task to deal with them. However, one can overcome this difficulty and remove it by using the Holstein–Primakoff transformation. These transformations are given by

$$\hat{J}_- = \sqrt{\hat{J} - \hat{n}_c}\,\hat{c}\,\mathrm{e}^{-\mathrm{i}\gamma(t)} \qquad \hat{J}_+ = \hat{c}^{\dagger}\sqrt{\hat{J} - \hat{n}_c}\,\mathrm{e}^{\mathrm{i}\gamma(t)} \qquad \hat{n} = \hat{c}^{\dagger}\hat{c} = \tfrac{1}{2}(\hat{J} + \hat{J}_z) \tag{1.3}$$

where $\gamma(t)$ is an arbitrary function of time. Under this transformation the Hamiltonian model changes from fermion–boson to boson–boson and now takes the form

$$\frac{\hat{H}(t)}{\hbar} = \omega\hat{a}^{\dagger}\hat{a} + \omega_0 \hat{c}^{\dagger}\hat{c} + \lambda\left(\hat{a}^{\dagger m}\sqrt{\hat{J} - \hat{n}_c}\,\hat{c}\,\mathrm{e}^{-\mathrm{i}\gamma(t)} + \hat{a}^m \hat{c}^{\dagger}\sqrt{\hat{J} - \hat{n}_c}\,\mathrm{e}^{\mathrm{i}\gamma(t)}\right). \tag{1.4}$$

Furthermore, if we invoke the transformation $\hat{a}^m = f(\hat{n}_a)\hat{b}$ and $\hat{a}^{\dagger m} = \hat{b}^{\dagger} f(\hat{n}_a)$, where the number operators $\hat{n}_a$ and $\hat{n}_b$ are connected by $\hat{n}_a = m\hat{n}_b$, the Hamiltonian (1.4) becomes

$$\frac{\hat{H}(t)}{\hbar} = m\omega\hat{b}^{\dagger}\hat{b} + \omega_0\hat{c}^{\dagger}\hat{c} + g(t)(\hat{b}^{\dagger}\hat{c}\,\mathrm{e}^{-\mathrm{i}\gamma(t)} + \hat{b}\hat{c}^{\dagger}\,\mathrm{e}^{\mathrm{i}\gamma(t)}) \tag{1.5}$$

where $g(t)$ is the coupling parameter given by

$$g(t) = \lambda\sqrt{\hat{J} - \hat{n}_c}\left[\frac{[m(\hat{n}_b + 1)]!}{(\hat{n}_b + 1)!(m\hat{n}_b)!}\right] \qquad \hat{J} \geqslant \hat{n}_c. \tag{1.6}$$

In equation (1.5) we have approximated $\hat{n}_c(t)$ and $\hat{n}_a(t)$ with their $C$-number time-dependent functions $n_c(t)$ and $n_a(t)$ by regarding $\hat{n}_c(t) - n_c$ and $\hat{n}_a(t) - n_a$ as small perturbations. Here we wish to emphasize that the operators $b(b^{\dagger})$ and $c(c^{\dagger})$ satisfy commutation relations similar to that of the operators $a(a^{\dagger})$. In fact the Hamiltonian (1.5) can be regarded as the most general time-dependent frequency converter model. Therefore, if we set $\omega_0 = -m\omega$, then the above Hamiltonian system represents the **SU**(2) Lie algebra and, in this case, one can use the direct properties of the Lie algebra to find the wavefunction for the transformed Hamiltonian. However, after some manipulations and without any restriction on the Hamiltonian model we have managed elsewhere to find the most general solution of the wavefunction using the Lie algebra technique [18]. Furthermore, the statistical properties of such systems have already been considered (for more details see [19]). In the following sections we study the constants of the motion related to the time-dependent Hamiltonian (1.5). For this reason we devote section 2 to considering the exact resonance case (absence of the detuning parameter). There are two main cases: linear and quadratic invariants, where in each case we introduce classes of both real and complex invariants. The more complicated case (off-resonance case) is treated in section 3, where the effect of the detuning parameter is examined. In section 4 we find the eigenvalues and the corresponding eigenfunctions for some constants of the motion which we have obtained. Finally we present our conclusion in section 5.

## 2. Exact resonance cases

In this section we restrict our treatment to the problem of constants of the motion in the exact resonance case. We seek linear and quadratic invariants and in each case both real and complex invariants are considered.

### 2.1. Linear invariants

*2.1.1. Real invariants.* It is more instructive to concentrate on finding the linear and the quadratic invariants for the Hamiltonian (1.5) in the case of off-resonance, i.e. where $\omega_0 \neq m\omega$. However, firstly we consider the fundamental first-degree invariants for this system at exact resonance in the absence of the time-dependent phase pump $\gamma(t)$. In this context we handle two kinds of invariants; the first is real while the second is complex. In the case of resonance and for $\gamma(t) = 0$, the Hamiltonian (1.5) reduces to

$$\frac{\hat{H}(t)}{\hbar} = \omega_0(\hat{b}^{\dagger}\hat{b} + \hat{c}^{\dagger}\hat{c}) + g(t)(\hat{b}^{\dagger}\hat{c} + \hat{b}\hat{c}^{\dagger}). \tag{2.1}$$

We define two pairs of Dirac operators in terms of the coordinates $\hat{q}_i$ and the momentum $\hat{p}_i$ such that

$$\hat{b} = \frac{(\omega_0\hat{q}_1 + \mathrm{i}\hat{p}_1)}{\sqrt{2\omega_0\hbar}} \qquad \hat{c} = \frac{(\omega_0\hat{q}_2 + \mathrm{i}\hat{p}_2)}{\sqrt{2\omega_0\hbar}}. \tag{2.2}$$

Then the Hamiltonian (2.1) takes the form

$$\hat{H}(t) = \frac{1}{2}\sum_{i=1}^{2}\left(\hat{p}_i^2 + \omega_0^2\hat{q}_i^2\right) + \frac{g(t)}{\omega_0}\left(\hat{p}_1\hat{p}_2 + \omega_0^2\hat{q}_1\hat{q}_2\right). \tag{2.3}$$

We now introduce a constant of motion $I^{(1)}(t)$ of the form

$$\hat{I}^{(1)}(t) = \sum_{i=1}^{2}(\alpha_i\hat{p}_i + \beta_i\hat{q}_i) \qquad i = 1, 2 \tag{2.4}$$

where $\alpha_i(t)$ and $\beta_i(t)$ are arbitrary functions of time. To establish that the operator $\hat{I}^{(1)}(t)$ is a constant of motion, we have to determine the explicit expressions for the functions $\alpha_i(t)$ and $\beta_i(t)$. This can be done if one uses the equation

$$\frac{\mathrm{d}\hat{I}^{(1)}}{\mathrm{d}t} = \frac{\partial \hat{I}^{(1)}}{\partial t} + \sum_{i=1}^{2} \frac{\partial \hat{I}^{(1)}}{\partial \hat{q}_i}\frac{\partial \hat{H}}{\partial \hat{p}_i} - \frac{\partial \hat{I}^{(1)}}{\partial \hat{p}_i}\frac{\partial \hat{H}}{\partial \hat{q}_i} = 0. \tag{2.5}$$

From equations (2.3) and (2.4), together with equation (2.5), and after some calculation we have

$$\alpha_{\underline{i}}(t) = (A_1[\cos(\omega_0 t + \phi_1) + k(t)] \pm A_2[\cos(\omega_0 t + \phi_2) - k(t)]) \qquad \underline{i} = 1, 2 \tag{2.6}$$

$$\beta_{\underline{i}}(t) = \omega_0(A_1[\sin(\omega_0 t + \phi_1) + k(t)] \pm A_2[\sin(\omega_0 t + \phi_2) - k(t)]) \qquad \underline{i} = 1, 2 \tag{2.7}$$

where $k(t) = \int_0^t g(t')\,\mathrm{d}t'$ and $A_{(1,2)}$ and $\phi_{(1,2)}$ are arbitrary constants expressible in terms of $\alpha_i(0)$ and $\beta_i(0)$. On the other hand the equations of motion in the Heisenberg picture for the Hamiltonian (2.3) can be written as

$$\begin{aligned} \frac{\mathrm{d}\hat{p}_1}{\mathrm{d}t} &= -\omega_0^2\hat{q}_1 - \omega_0 g(t)\hat{q}_2 & \qquad \frac{\mathrm{d}\hat{p}_2}{\mathrm{d}t} &= -\omega_0^2\hat{q}_2 - \omega_0 g(t)\hat{q}_1 \\ \frac{\mathrm{d}\hat{q}_1}{\mathrm{d}t} &= \hat{p}_1 + \frac{g(t)}{\omega_0}\hat{p}_2 & \qquad \frac{\mathrm{d}\hat{q}_2}{\mathrm{d}t} &= \hat{p}_2 + \frac{g(t)}{\omega_0}\hat{p}_1 \end{aligned} \tag{2.8}$$

which can be written in the more attractive form

$$\dot{\hat{\mathbf{q}}} = \frac{1}{\omega_0}\begin{pmatrix} \omega_0 & g \\ g & \omega_0 \end{pmatrix}\hat{\mathbf{p}} \qquad \dot{\hat{\mathbf{p}}} = -\omega_0\begin{pmatrix} \omega_0 & g \\ g & \omega_0 \end{pmatrix}\hat{\mathbf{q}} \tag{2.9}$$

and their solutions are

$$\begin{aligned} \hat{q}_1(t) &= B_1\sin[(\omega_0 t + \psi_1) - k(t)] + B_2\sin[(\omega_0 t + \psi_2) + k(t)] \\ \hat{q}_2(t) &= B_2\sin[(\omega_0 t + \psi_2) + k(t)] - B_1\sin[(\omega_0 t + \psi_1) - k(t)] \\ \hat{p}_1(t) &= \omega_0[B_1\cos[(\omega_0 t + \psi_1) - k(t)] + B_2\cos[(\omega_0 t + \psi_2) + k(t)]] \\ \hat{p}_2(t) &= \omega_0[B_2\cos[(\omega_0 t + \psi_2) + k(t)] - B_1\cos[(\omega_0 t + \psi_1) - k(t)]] \end{aligned} \tag{2.10}$$

where $B_{1,2}$ are arbitrary constants.

Thus, if one substitutes equations (2.6) and (2.8) into equation (2.4), one has

$$\hat{I}^{(1)}(t) = 2\omega_0[A_1B_2\cos(\phi_1 - \psi_2) + A_2B_1\cos(\phi_2 - \psi_1)] \tag{2.11}$$

which is constant.

On the other hand one may think of different forms for the linear invariant of the same system. To see that we define the following operators:

$$\hat{p}_\pm = \frac{\hat{p}_1 \pm \hat{p}_2}{\sqrt{2}} \tag{2.10$a$}$$

$$\hat{q}_\pm = \frac{\hat{q}_1 \pm \hat{q}_2}{\sqrt{2}} \tag{2.10$b$}$$

with the properties $[\hat{q}_\pm, \hat{p}_\pm] = \mathrm{i}\hbar\delta_\pm$, where $\delta_\pm$ is either one if the signs are the same or zero otherwise. If we substitute equation (2.10) into equation (2.3), we find

$$\hat{H}(t) = \tfrac{1}{2}\left[G_+(t)\left(\hat{p}_+^2 + \omega_0^2\hat{q}_+^2\right) + G_-(t)\left(\hat{p}_-^2 + \omega_0^2\hat{q}_-^2\right)\right] \tag{2.12}$$

where $G_{\pm}(t) = (1 \pm g(t)/\omega_0)$. Now, if we introduce the transformation $\hat{P}_{(1,2)}(t) = \sqrt{G_{\pm}(t)}\hat{p}_{\pm}(t)$ and $\hat{Q}_{(1,2)}(t) = 1/\sqrt{G_{\pm}(t)}\hat{q}_{\pm}(t)$, the Hamiltonian ( 2.12) is transformed to

$$\hat{H}(t) = \frac{1}{2}\sum_{i=1}^{2}\left[\hat{P}_i^2 + \Omega_i^2(t)\hat{Q}_i^2 + \varepsilon_i(t)\hat{P}_i\hat{Q}_i\right] \tag{2.13}$$

where

$$[\hat{Q}_i, \hat{P}_i] = \mathrm{i}\hbar\delta_{ij} \qquad \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{2.14}$$

while $\Omega_i(t) = (\omega_0 \pm g(t))$ and $\varepsilon_i(t) = -(\mathrm{d}\ln\Omega_i/\mathrm{d}t)$.

Since we assume that the commutation relation (2.14) holds, equation (2.13) is simply identified as the linear superposition of two independent oscillators. Thus from equations (2.4) and (2.5) together with equation (2.13) we obtain the following:

$$\frac{\mathrm{d}\alpha_i}{\mathrm{d}t} + \beta_i = \frac{1}{2}\varepsilon_i(t)\alpha_i \qquad \frac{\mathrm{d}\beta_i}{\mathrm{d}t} + \frac{1}{2}\varepsilon_i(t)\beta_i = \Omega_i^2(t)\alpha_i. \tag{2.15}$$

Eliminating $\beta_i$ we find that

$$\frac{\mathrm{d}^2\alpha_i}{\mathrm{d}t^2} + \Omega_{Q_i}^2(t)\alpha_i = 0 \qquad \Omega_{Q_i}(t) = \left(\Omega_i^2(t) - \frac{1}{2}\varepsilon_i' - \frac{1}{4}\varepsilon_i^2\right)^{1/2}. \tag{2.16}$$

We may denote any solution of equation (2.16) by $\alpha_i(t) = \sigma_i^{(0)}(t)$. Then the invariant assumes the form

$$\hat{I}^{(Q)}(t) = \sum_{i=1}^{2}\left[\sigma_i^{(0)}(t)\hat{P}_i + \left(\frac{\varepsilon_i}{2}\sigma_i^{(0)}(t) - \frac{\mathrm{d}\sigma_i^{(0)}(t)}{\mathrm{d}t}\right)\hat{Q}_i\right]. \tag{2.17}$$

Eliminating $\alpha_i(t)$ we find that

$$\frac{\mathrm{d}^2\beta_i}{\mathrm{d}t^2} + 2\varepsilon_i\frac{\mathrm{d}\beta_i}{\mathrm{d}t} + \Omega_{P_i}^2(t)\beta_i = 0 \qquad \Omega_{P_i}(t) = \left(\Omega_i^2(t) + \frac{1}{2}\varepsilon_i' + \frac{3}{4}\varepsilon_i^2\right)^{1/2}. \tag{2.18}$$

If we denote by $\rho_i^{(0)}(t) = \beta_i(t)$ any solution of equation (2.18), then the invariant takes the form

$$\hat{I}^{(P)}(t) = \sum_{i=1}^{2}\left[\rho_i^{(0)}(t)\hat{Q}_i + \left(\frac{\varepsilon_i}{2}\rho_i^{(0)}(t) + \frac{\mathrm{d}\rho_i^{(0)}(t)}{\mathrm{d}t}\right)\hat{P}_i \Big/ \Omega_i^2(t)\right]. \tag{2.19}$$

Equations (2.17) and (2.19) represent two classes of linear constants of the motion.

*2.1.2. Complex invariants.* In order to obtain a complex invariant for the present system, we use equation (2.10). Thus, after we rewrite $B_{(1,2)}$ and $\psi_{(1,2)}$ in terms of $\hat{J}_{(1,2)}$ and $\hat{I}_{(1,2)}$ and make use of the inverse transformations, we obtain the complex invariants in the form

$$\begin{aligned} \hat{C}_1(t) &= \omega_0(\hat{I}_2 + \mathrm{i}\hat{I}_1) = \tfrac{1}{2}\left[\omega_0(\hat{q}_1 + \hat{q}_2) + \mathrm{i}(\hat{p}_1 + \hat{p}_2)\right]\exp\{\mathrm{i}(\omega_0 t + k(t))\} \\ \hat{C}_2(t) &= \omega_0(\hat{J}_2 + \mathrm{i}\hat{J}_1) = \tfrac{1}{2}\left[\omega_0(\hat{q}_1 - \hat{q}_2) + \mathrm{i}(\hat{p}_1 - \hat{p}_2)\right]\exp\{\mathrm{i}(\omega_0 t - k(t))\} \end{aligned} \tag{2.20}$$

where the $\hat{C}_i(t)$, $i = 1, 2$, with their complex conjugates are Dirac variables with evolution in negative time, which evaluates them at a previous time $t = 0$.

A second-degree energy-like invariant is

$$|\hat{C}_+(t)|^2 = \left[\left(\hat{p}_1^2 + \omega_0^2\hat{q}_1^2\right)\cos^2 k(t) + \left(\hat{p}_2^2 + \omega_0^2\hat{q}_2^2\right)\sin^2 k(t) + \omega_0\left(\hat{q}_2\hat{p}_1 - \hat{q}_1\hat{p}_2\right)\sin 2k(t)\right] \tag{2.21}$$

where $\hat{C}_+(t) = \hat{C}_1(t) + \hat{C}_2(t)$. Alternatively we may construct another constant of the motion, such as $\hat{C}_-(t) = \hat{C}_1(t) - \hat{C}_2(t)$, in which case we find

$$|\hat{C}_-(t)|^2 = \left[\left(\hat{p}_1^2 + \omega_0^2\hat{q}_1^2\right)\sin^2 k(t) + \left(\hat{p}_2^2 + \omega_0^2\hat{q}_2^2\right)\cos^2 k(t) - \omega_0\left(\hat{q}_2\hat{p}_1 - \hat{q}_1\hat{p}_2\right)\sin 2k(t)\right]. \tag{2.22}$$

Note that, from the above equations and at $t = 0$, we have no interaction between the modes. Then the constant of the motion in each case is just a free harmonic oscillator. This is in agreement with the fundamental basis of the theory of the interaction between the modes.

*2.1.3. Quadratic invariants.* We continue our progress and turn our attention to consider a quadratic invariant. To introduce a second-degree invariant $\hat{I}^{(2)}(t)$ in the form

$$\hat{I}^{(2)}(t) = \sum_{i=1}^{2}\left[\bar{\alpha}_i(t)\hat{q}_i^2 + \bar{\beta}_i(t)\hat{p}_i^2 + 2\bar{\gamma}_i(t)\hat{q}_i\hat{p}_i\right] + (\mu_1\hat{q}_1\hat{q}_2 + \mu_2\hat{p}_1\hat{p}_2 + \mu_3\hat{q}_1\hat{p}_2 + \mu_4\hat{p}_1\hat{q}_2) \tag{2.23}$$

together with equations (2.3) and (2.5) would lead to the complicated situation in which we would have to solve ten simultaneous differential equations. However, to avoid this complication we deal with the diagonalized Hamiltonian (2.13) together with the equation

$$\hat{I}^{(2)}(t) = \sum_{i=1}^{2}\left[\bar{\alpha}_i(t)\hat{Q}_i^2 + \bar{\beta}_i(t)\hat{P}_i^2 + 2\bar{\gamma}_i(t)\hat{Q}_i\hat{P}_i\right]. \tag{2.24}$$

In this case we have

$$\frac{\mathrm{d}\bar{\alpha}_i}{\mathrm{d}t} + \varepsilon_i(t)\bar{\alpha}_i = 2\Omega_i^2(t)\bar{\gamma}_i \qquad \frac{\mathrm{d}\bar{\beta}_i}{\mathrm{d}t} - \varepsilon_i(t)\bar{\beta}_i = -2\bar{\gamma}_i \qquad \frac{\mathrm{d}\bar{\gamma}_i}{\mathrm{d}t} - \Omega_i^2(t)\bar{\beta}_i = -\bar{\alpha}_i. \tag{2.25}$$

Now, if we set $\bar{\beta}_i = k_i^{1/2}\sigma_i^2(t)$, where $k_i$ are some constants, then after some calculations we can express the invariant in terms of an auxiliary function $\sigma_i(t)$ that satisfies the Pinney equation [20]

$$\frac{\mathrm{d}^2\sigma_i}{\mathrm{d}t^2} + \Omega_{Q_i}^2(t)\sigma_i = \frac{1}{\sigma_i^3} \qquad \Omega_{Q_i}(t) = \left(\Omega_i^2(t) - \frac{1}{2}\dot{\varepsilon}_i - \frac{1}{4}\varepsilon_i^2\right)^{1/2}. \tag{2.26}$$

Thus

$$\hat{I}^{(Q)}(t) = \sum_{i=1}^{2} k_i^{1/2}\left[\hat{Q}_i^2/\sigma_i^2 + \{\sigma_i\hat{P}_i + (\sigma_i\varepsilon_i(t)/2 - \dot{\sigma}_i)\hat{Q}_i\}^2\right]. \tag{2.27}$$

Similarly, if we take $\bar{\alpha}_i = k_i^{1/2}\rho_i^2(t)$, then we obtain another form for the invariant as

$$\hat{I}^{(P)}(t) = \sum_{i=1}^{2} k_i^{1/2}\left[\hat{P}_i^2/\rho_i^2 + \left\{\rho_i\hat{Q}_i + (\rho_i\varepsilon_i(t)/2 + \dot{\rho}_i)\hat{P}_i/\Omega_i^2\right\}^2\right] \tag{2.28}$$

where $\rho_i(t)$ satisfies the equation

$$\frac{\mathrm{d}^2\rho_i}{\mathrm{d}t^2} + 2\varepsilon_i(t)\frac{\mathrm{d}\rho_i}{\mathrm{d}t} + \Omega_{p_i}^2\rho_i = \frac{\Omega_i^4}{\rho_i^3} \qquad \Omega_{p_i}^2 = \left(\Omega_i^2 + \frac{1}{2}\dot{\varepsilon}_i + \frac{3}{4}\varepsilon_i^2\right)^{1/2}. \tag{2.29}$$

Equations (2.26) and (2.29) are nonlinear second-order differential equations with variable coefficients. Their solutions after some manipulations take the forms

$$\begin{aligned}
\sigma_1(t) &= \left(\Omega_1(t)k_1^{1/2}\right)^{-1/2}[\tilde{k}_1 \pm a_1 \sin[2(\omega_0 t + k(t)) + \bar{\phi}_1]]^{1/2} \\
\rho_1(t) &= \left(\Omega_1(t)/k_1^{1/2}\right)^{1/2}[\tilde{k}_1 \mp a_1 \sin[2(\omega_0 t + k(t)) + \bar{\phi}_1]]^{1/2} \\
\sigma_2(t) &= \left(\Omega_2(t)k_2^{1/2}\right)^{-1/2}[\tilde{k}_2 \pm a_2 \sin[2(\omega_0 t - k(t)) + \bar{\phi}_2]]^{1/2} \\
\rho_2(t) &= \left(\Omega_2(t)/k_2^{1/2}\right)^{1/2}[\tilde{k}_2 \mp a_2 \sin[2(\omega_0 t - k(t)) + \bar{\phi}_2]]^{1/2}
\end{aligned} \tag{2.30}$$

where $a_i, k_i, \tilde{k}_i$ and $\bar{\phi}_i, i = 1, 2$ are arbitrary constants and phases respectively. Thus, if one uses the inverse transformation (2.10$a$), (2.10$b$), then an exact expression for the desired invariants can be obtained.

As a result of the conditions we have put on the Hamiltonian (1.5) ($\gamma(t) = 0$ and $m\omega = \omega_0$) our treatment of the problem so far is simple. This is because the Hamiltonian itself in this case is separable (see equations (2.12) and (2.13)).

Finally we may point out that the linear invariants (2.20) give rise to a pair of autonomous invariants, i.e. the first integrals

$$\begin{aligned}
\hat{J}_1 &= \hat{C}_1(t)\hat{C}_1^\dagger(t) \\
&= \tfrac{1}{4}\left[\omega_0^2(\hat{q}_1 + \hat{q}_2)^2 + (\hat{p}_1 + \hat{p}_2)^2\right]
\end{aligned} \tag{2.31}$$

$$\begin{aligned}
\hat{J}_2 &= \hat{C}_2(t)\hat{C}_2^\dagger(t) \\
&= \tfrac{1}{4}\left[\omega_0^2(\hat{q}_1 - \hat{q}_2)^2 + (\hat{p}_1 - \hat{p}_2)^2\right]
\end{aligned} \tag{2.32}$$

so that we obtain the separable first integral

$$\hat{J}_3 = \tfrac{1}{2}\left[\omega_0^2\left(\hat{q}_1^2 + \hat{q}_2^2\right) + \left(\hat{p}_1^2 + \hat{p}_2^2\right)\right] \tag{2.33}$$

and the solution of

$$\hat{J}_3\psi(t) = \mathrm{i}\hbar\frac{\partial\psi(t)}{\partial t} \tag{2.34}$$

follows trivially.

In the following sections we deal with the problem without any restrictions. This means that we treat the case in which the field's frequency $m\omega \neq \omega_0$ (off-resonance case). The purpose is to find the linear and quadratic invariants and also to employ the results to obtain the exact expression for the wavefunction in the coherent states representation.

## 3. Off-resonance case

We devote this section to considering the more complicated case of finding the linear and quadratic invariants for the Hamiltonian (1.5). In this case we deal with the problem in the presence of the detuning parameter effect (off-resonance case). To do so we firstly diagonalize the Hamiltonian (1.5). This is considered in the following subsection.

### *3.1. Diagonalized Hamiltonian*

In order to diagonalize the Hamiltonian (1.5) we firstly modify equation (2.2) and rewrite the operators $\hat{b}$ and $\hat{c}$ as

$$\hat{b} = \frac{(\bar{\omega}\hat{q}_1 + \mathrm{i}\hat{p}_1)}{\sqrt{2\bar{\omega}\hbar}} \qquad \hat{c} = \frac{(\omega_0\hat{q}_2 + \mathrm{i}\hat{p}_2)}{\sqrt{2\omega_0\hbar}} \tag{3.1}$$

where we set $\bar{\omega} = m\omega$.

Now, if we substitute equation (3.1) into equation (1.5), we obtain

$$\hat{H}(t) = \frac{1}{2}\sum_{i=1}^{2}\left(\hat{p}_i^2 + \omega_i^2\hat{q}_i^2\right) + g(t)\left[\left(\sqrt{\omega_1\omega_2}\hat{q}_1\hat{q}_2 + \frac{\hat{p}_1\hat{p}_2}{\sqrt{\omega_1\omega_2}}\right)\cos\gamma(t)\right.$$
$$\left. + \left(\sqrt{\frac{\omega_1}{\omega_2}}\hat{q}_1\hat{p}_2 - \sqrt{\frac{\omega_2}{\omega_1}}\hat{q}_2\hat{p}_1\right)\sin\gamma(t)\right] \tag{3.2}$$

where we have taken for convenience $\bar{\omega} = \omega_1$ and $\omega_0 = \omega_2$. We introduce the canonical transformations

$$\begin{aligned}
\sqrt{\omega_1}\hat{q}_1 &= \bar{Q}_1\cos\delta_+(t) + \bar{P}_1\sin\delta_+(t)\\
\hat{p}_1/\sqrt{\omega_1} &= \bar{P}_1\cos\delta_+(t) - \bar{Q}_1\sin\delta_+(t)\\
\sqrt{\omega_2}\hat{q}_2 &= \bar{Q}_2\cos\delta_-(t) + \bar{P}_2\sin\delta_-(t)\\
\hat{p}_2/\sqrt{\omega_2} &= \bar{P}_2\cos\delta_-(t) - \bar{Q}_2\sin\delta_-(t)
\end{aligned} \tag{3.3}$$

where the time-dependent angles, $\delta_\pm(t)$, are given by

$$\delta_\pm(t) = \tfrac{1}{2}[(\omega_1 + \omega_2)t \pm \gamma(t)]. \tag{3.4}$$

Thus one can find that the Hamiltonian (3.2) reduces to

$$\hat{H}(t) = \frac{\Delta(t)}{2}\left[\left(\bar{P}_1^2 + \bar{Q}_1^2\right) - \left(\bar{P}_2^2 + \bar{Q}_2^2\right)\right] - g(t)[\bar{Q}_1\bar{Q}_2 + \bar{P}_1\bar{P}_2] \tag{3.5}$$

where

$$\Delta(t) = \tfrac{1}{2}(\dot{\gamma}(t) + \omega_2 - \omega_1) \tag{3.6}$$

and an overdot indicates the derivative with respect to time. Furthermore, if we invoke the transformations

$$\begin{bmatrix}\bar{Q}_1\\ \bar{Q}_2\\ \bar{P}_1\\ \bar{P}_2\end{bmatrix} = \begin{bmatrix}\cos\delta & -\sin\delta & 0 & 0\\ \sin\delta & \cos\delta & 0 & 0\\ 0 & 0 & \cos\delta & -\sin\delta\\ 0 & 0 & \sin\delta & \cos\delta\end{bmatrix}\begin{bmatrix}\hat{x}\\ \hat{y}\\ \hat{p}_x\\ \hat{p}_y\end{bmatrix} \tag{3.7}$$

then the Hamiltonian (3.5) reduces immediately to the form

$$\hat{H}(t) = \tfrac{1}{2}S(t)\left\{\left[\hat{p}_x^2 + \hat{x}^2\right] - \left[\hat{p}_y^2 + \hat{y}^2\right]\right\} + \dot{\delta}(t)(\hat{x}\hat{p}_y - \hat{y}\hat{p}_x) \tag{3.8}$$

where

$$\delta(t) = \frac{1}{2}\tan^{-1}\left(\frac{g(t)}{\Delta(t)}\right) \tag{3.9}$$

and $S(t) = \sqrt{\Delta^2(t) + g^2(t)}$. Now, if we take $g(t)/\Delta(t) = \text{const}$, then $\dot{\delta}(t) = 0$ and hence the Hamiltonian (3.8) is immediately diagonalized.

It should be noted that in previous calculations we have taken care of the generating function completely during the applications of the canonical transformations (see [21]). The condition $g(t)/\Delta(t) = \text{const}$ corresponds to the integrability condition used by Lu [22] which enabled him to study the problem of coupled oscillators. Alternatively we may introduce another integrability condition more general than the previous one, that is

$$\dot{\delta}(t) = \Gamma\sqrt{\Delta^2(t) + g^2(t)} \tag{3.10}$$

where $\Gamma$ is some constant. In this case the Hamiltonian (3.8) under the transformation

$$\begin{bmatrix}\hat{x}\\ \hat{y}\\ \hat{p}_x\\ \hat{p}_y\end{bmatrix} = \begin{bmatrix}\cos\theta & 0 & 0 & \sin\theta\\ 0 & \cos\theta & \sin\theta & 0\\ 0 & -\sin\theta & \cos\theta & 0\\ -\sin\theta & 0 & 0 & \cos\theta\end{bmatrix}\begin{bmatrix}\hat{x}_1\\ \hat{y}_1\\ \hat{p}_{x_1}\\ \hat{p}_{y_1}\end{bmatrix} \tag{3.11}$$

where $\theta = \frac{1}{2}\tan^{-1}\Gamma$, becomes diagonal and hence takes the form

$$\hat{H}(t) = \tfrac{1}{2}\sqrt{(1+\Gamma^2)(\Delta^2(t)+g^2(t))}\left[\left(\hat{p}_{x_1}^2+\hat{x}_1^2\right)-\left(\hat{p}_{y_1}^2+\hat{y}_1^2\right)\right]. \tag{3.12}$$

On the other hand one can diagonalize the Hamiltonian (1.5) and rewrite it in terms of the creation and annihilation operators. This can be done if we define two pairs of Dirac operators as

$$\hat{A}_1(t) = \left[\hat{b}[\cos\theta\cos\delta(t) - \mathrm{i}\sin\theta\sin\delta(t)]\exp\left(\frac{\mathrm{i}}{2}\gamma(t)\right)\right] + \left[\hat{c}[\cos\theta\sin\delta(t) + \mathrm{i}\sin\theta\cos\delta(t)]\exp\left(-\frac{\mathrm{i}}{2}\gamma(t)\right)\right] \tag{3.13}$$

and

$$\hat{A}_2(t) = \left[\hat{c}[\cos\theta\cos\delta(t) + \mathrm{i}\sin\theta\sin\delta(t)]\exp\left(-\frac{\mathrm{i}}{2}\gamma(t)\right)\right] - \left[\hat{b}[\cos\theta\sin\delta(t) - \mathrm{i}\sin\theta\cos\delta(t)]\exp\left(\frac{\mathrm{i}}{2}\gamma(t)\right)\right]. \tag{3.14}$$

Now, if we substitute equations (3.13) and (3.14) into equation (1.5) and apply the integrability condition (3.10), then after some calculation we have

$$\frac{\hat{H}(t)}{\hbar} = \dot{\mu}_+(t)\left(\hat{A}_1^\dagger\hat{A}_1 + \frac{1}{2}\right) + \dot{\mu}_-(t)\left(\hat{A}_2^\dagger\hat{A}_2 + \frac{1}{2}\right) \tag{3.15}$$

where

$$\mu_\pm(t) = \left[\tfrac{1}{2}(\omega_1+\omega_2)t \pm (\delta(t)-\delta(0))\,\mathrm{cosec}\,2\theta\right]. \tag{3.16}$$

Here we may point out that, although the integrability condition we have introduced is more general than that of Lu [22], both can be used to diagonalize the Hamiltonian. However, in principle the integrability conditions would restrict the results and therefore we have to deal with the unrestricted condition to obtain the most general form of the constants of the motion. Due to the nature of the present case of two time-dependent coupled oscillators we find that it is unlikely to avoid any use of the integrability condition.

Despite our doubtful opinion of the feasibility of diagonalizing (3.8 ) in general, we can make some progress by using the theory of time-dependent linear canonical transformations [23]. In the case of a quadratic Hamiltonian

$$\hat{H} = \tfrac{1}{2}\hat{\mathbf{z}}^{\mathrm{T}}A\hat{\mathbf{z}} \tag{3.17}$$

where $\hat{z}^\mu = \hat{q}^i$, $\mu = 1, n$; $i = 1, n$ and $\hat{z}^\mu = \hat{p}^i$, $\mu = n+1, 2n$; $i = 1, n$, and $A$ is a $2n \times 2n$ Hermitian matrix, we recall that under the linear canonical transformation

$$\hat{\bar{\mathbf{z}}} = W\hat{\mathbf{z}} \qquad W\tilde{J}W^{\mathrm{T}} = \tilde{J} \tag{3.18}$$

where $W$ is a $2n \times 2n$ matrix with possibly time-dependent elements and $\tilde{J}$ is the $2n \times 2n$ symplectic matrix, the transformed Hamiltonian is

$$\bar{H} = \tfrac{1}{2}\hat{\bar{\mathbf{z}}}\bar{A}\hat{\bar{\mathbf{z}}} \tag{3.19}$$

and the matrix of the transformation is given by the solution of the homogeneous linear system

$$\dot{W} = \tilde{J}\bar{A}W - W\tilde{J}A. \tag{3.20}$$

Furthermore, provided the $(2n)^2$ arbitrary constants arising from the integration of (3.20) are chosen such that the transformation is canonical, i.e. $W(t_0)\tilde{J}W(t_0)^{\mathrm{T}} = \tilde{J}$, at some $t_0$, the transformation is canonical for all time such that the integration of (3.20) remains valid, i.e.

for the interval of time containing $t_0$ such that the elements of $A$ and $\bar{A}$ are continuous functions of time [24] (p 72).

The number of arbitrary constants arising from the integration of (3.20) is $(2n)^2$ and the number of constraints imposed by the requirement of canonicity is $n(2n-1)$. Consequently there is still choice in the selection of the constants to make the transformation simpler than indicated by the general solution of (3.20) [25].

The choice of the transformed Hamiltonian is dictated by considerations of ease of solution of the consequent Heisenberg equations of motion or the Schrödinger equation subject to a modicum of commonsense. The structure of $\bar{A}$ should reflect that of $A$ so that $\hat{H}$ and $\bar{H}$ describe systems which are qualitatively the same[3]. In the case of a one-degree-of-freedom problem it is usual to choose $W$ such that the canonical transformation is a point transformation. The price to pay is that $\bar{A}$ cannot be taken to be autonomous. The classic example is that of the time-dependent harmonic oscillator described by [1, 2]

$$\hat{H} = \tfrac{1}{2}(\hat{p}^2 + \omega^2(t)\hat{q}^2). \tag{3.21}$$

Under the time-dependent linear point transformation [27]

$$\hat{Q} = \frac{\hat{q}}{\rho(t)} \qquad \hat{P} = \rho\hat{p} - \dot{\rho}\hat{q} \tag{3.22}$$

where $\rho(t)$ is a solution of the famous Ermakov–Pinney equation [29, 20]

$$\ddot{\rho}(t) + \omega^2(t)\rho(t) = \frac{1}{\rho^3(t)} \tag{3.23}$$

we obtain

$$\bar{H} = \frac{1}{2\rho^2(t)}(\hat{P}^2 + \hat{Q}^2). \tag{3.24}$$

Since the time dependence of $\bar{A}$ is multiplicative, the change of time variable to $T := \int \rho^{-2}\,\mathrm{d}t$ brings one to the equivalent autonomous system

$$\tilde{H} = \tfrac{1}{2}(\hat{P}^2 + \hat{Q}^2) \tag{3.25}$$

which is readily solved.

The convenience of the choice of a point transformation does not persist for systems of more than one degree of freedom. We must treat the general linear canonical transformation. However, there is some compensation in that $\bar{A}$ may be chosen to be a constant matrix [28], i.e. there is no need for the introduction of new time.

In the case of (3.8)

$$A = \begin{bmatrix} S(t) & 0 & 0 & \dot{\delta}(t) \\ 0 & -S(t) & -\dot{\delta}(t) & 0 \\ 0 & -\dot{\delta}(t) & S(t) & 0 \\ \dot{\delta}(t) & 0 & 0 & -S(t) \end{bmatrix} = \begin{bmatrix} S(t)K & \dot{\delta}(t)J \\ \dot{\delta}(t)J^T & S(t)K \end{bmatrix} \tag{3.26}$$

where $J$ is the $2 \times 2$ symplectic matrix and

$$K := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{3.27}$$

We note that, since $\det A = (S^2 + \dot{\delta}^2)^2$, the matrix $A$ is positive definite for all real $S(t)$ and $\dot{\delta}(t)$ not simultaneously zero.

[3] In classical mechanics this is not a problem, but it becomes critical in quantal systems [26].

A suitable candidate for $\bar{A}$ is

$$\bar{A} = \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix} \tag{3.28}$$

for which (3.20) becomes

$$\begin{aligned} \dot{W}_1 &= \dot{\delta} W_1 \tilde{J} + S W_2 K + K W_3 \\ \dot{W}_2 &= -S W_1 K + \dot{\delta} W_2 \tilde{J} + K W_4 \\ \dot{W}_3 &= \dot{\delta} W_3 \tilde{J} + S W_4 K - K W_1 \\ \dot{W}_4 &= -S W_3 K + \dot{\delta} W_4 \tilde{J} - K W_2 \end{aligned} \tag{3.29}$$

where we have written

$$W = \begin{bmatrix} W_1 & W_2 \\ W_3 & W_4 \end{bmatrix}. \tag{3.30}$$

For general functions $S(t)$ and $\dot{\delta}(t)$ the solution of (3.29) is not transparent and normally we would have to resort to numerical methods. However, we do note that we have a formal definition of the canonical transformation of $\hat{H}$ (3.8) to

$$\bar{H} = \tfrac{1}{2}\left[\left(\hat{P}_X^2 + \hat{X}^2\right) - \left(\hat{P}_Y^2 + \hat{Y}^2\right)\right] \tag{3.31}$$

which is a system we may solve explicitly. In particular we have the creation and annihilation operators. Furthermore under the transformation (3.18) we have the explicit expressions for the invariant $\bar{H}$ and the creation and annihilation operators in terms of the variables of (3.8) and the elements of $W(t)$ which are simply scalar functions of time. Consequently we may make all formal operations with these operators with results which depend on time functions the explicit expressions for which require the solution of (3.29) subject to (3.18*b*). Numerical integration of (3.29) enables the final result to be expressed in a numerically useful form [30] .

### *3.2. Linear invariants*

Similarly to that given by equation (2.4) we define a linear invariant $\hat{J}^{(1)}(t)$ in the form

$$\hat{J}^{(1)}(t) = \sum_{i=1}^{2} (\mu_i(t)\hat{p}_i + \nu_i(t)\hat{q}_i). \tag{3.32}$$

If we substitute equation (3.1) into equation (3.32), we can rewrite the linear invariant in the form

$$\hat{J}^{(1)}(t) = \sum_{j=1}^{2} \left(\bar{\lambda}_j(t)\hat{a}_j + \bar{\lambda}_j^*(t)\hat{a}_j^\dagger\right) \tag{3.33}$$

where we have taken for convenience $b = \hat{a}_1$ and $c = \hat{a}_2$, while $\bar{\lambda}_i$ is given by

$$\bar{\lambda}_j(t) = \sqrt{\frac{\hbar}{2\omega_j}}[\nu_j(t) - \mathrm{i}\omega_j\mu_j(t)] \qquad j = 1, 2. \tag{3.34}$$

Thus, from equations (1.5), (3.33) and (3.34) together with equation (2.5), we obtain

$$\frac{\mathrm{d}\bar{\lambda}_1}{\mathrm{d}t} = \mathrm{i}\omega_1\bar{\lambda}_1 + \mathrm{i}g\,\mathrm{e}^{\mathrm{i}\gamma(t)}\bar{\lambda}_2 \qquad \frac{\mathrm{d}\bar{\lambda}_2}{\mathrm{d}t} = \mathrm{i}\omega_2\bar{\lambda}_2 + \mathrm{i}g\,\mathrm{e}^{-\mathrm{i}\gamma(t)}\bar{\lambda}_1. \tag{3.35}$$

In the framework of the integrability condition (3.10) the above coupled equations can be solved and hence the constants of the motion are determined. However, we can find the solution for some other special cases. For example, if we consider the case in which the phase pump $\gamma(t) = (\omega_1 - \omega_2)t$, then we have

$$\begin{aligned} \nu_i(t) &= a_i \cos(k(t) + \omega_i t) + b_i \cos(k(t) - \omega_i t) \\ \mu_i(t) &= b_i \sin(k(t) - \omega_i t) - a_i \sin(k(t) + \omega_i t) \qquad i = 1, 2 \end{aligned} \tag{3.36}$$

where $a_1 = a_2$ and $b_1 = -b_2$.

Moreover, and under the restrictive integrability condition (3.10), the general solution of equation (3.35) is given by

$$\begin{aligned} \bar{\lambda}_1(t) &= \mathrm{e}^{\mathrm{i}\omega_+(t)}[l \cos\delta(t) \cos(\eta_1 + (\delta(t) - \delta(0)) \operatorname{cosec} 2\theta) \\ &\qquad - j \sin\delta(t) \sin(\eta_2 + (\delta(t) - \delta(0)) \operatorname{cosec} 2\theta)] \\ \bar{\lambda}_2(t) &= \mathrm{e}^{-\mathrm{i}\omega_-(t)}[j \cos\delta(t) \sin(\eta_1 + (\delta(t) - \delta(0)) \operatorname{cosec} 2\theta) \\ &\qquad + l \sin\delta(t) \cos(\eta_2 + (\delta(t) - \delta(0)) \operatorname{cosec} 2\theta)] \end{aligned} \tag{3.37}$$

where $\omega_\pm(t) = [(\delta(t) - \delta(0)) \pm (\omega_1 + \omega_2)t]$, while $l$, $j$ and $\eta_i$ are arbitrary constants and phases respectively. Thus from equations (3.34) and (3.37) we can easily obtain the explicit expression for both $\nu_j$ and $\mu_j$. Hence the desired result can be found.

### 3.3. *Quadratic invariants*

Now we turn our attention to consider the quadratic invariants for the off-resonance case. The situation in this case is more complicated than that which we have studied before. This is in fact due to the complexity of equation (2.23).

*3.3.1. Real invariants.* We start this subsection by considering the real invariant for the Hamiltonian (1.5) under the restriction of the integrability condition (3.10). For this purpose we use the diagonalized Hamiltonian (3.12) together with the invariant given by equation (2.24). However, we firstly rewrite this Hamiltonian in the form

$$\begin{aligned} H(t) \to \bar{H}(t) = \frac{1}{2}\Bigg[\Bigg\{\bar{P}_x^2 + \eta^2(t)\bar{X}^2 - \frac{\dot{\eta}}{2\eta}(\bar{X}\bar{P}_x + \bar{P}_x\bar{X})\Bigg\} \\ - \Bigg\{\bar{P}_y^2 + \eta^2(t)\bar{Y}^2 + \frac{\dot{\eta}}{2\eta}(\bar{Y}\bar{P}_y + \bar{P}_y\bar{Y})\Bigg\}\Bigg] \end{aligned} \tag{3.38}$$

where we have taken $\eta(t) = \sqrt{(1 + \Gamma^2)(\Delta^2(t) + g^2(t))}$, define $x_1 = \sqrt{\eta(t)}\bar{X}$ and $\bar{P}_x = \sqrt{\eta(t)}p_{x_1}$. Also we defined $y_1 = \sqrt{\eta(t)}\bar{Y}$ and $\bar{P}_y = \sqrt{\eta(t)}p_{y_1}$. In this case we obtain the following invariants:

$$I^{(p)}(t) = \left[\left\{\left(\bar{\sigma}_1 x_1 + \frac{\dot{\bar{\sigma}}_1}{\eta}p_{x_1}\right)^2 + \frac{C_1}{\eta\bar{\sigma}_1^2}p_{x_1}^2\right\} + \left\{\left(\bar{\sigma}_2 y_1 - \frac{\dot{\bar{\sigma}}_2}{\eta}p_{y_1}\right)^2 + \frac{C_2}{\eta\bar{\sigma}_2^2}p_{y_1}^2\right\}\right] \tag{3.39}$$

and

$$I^{(q)}(t) = \left[\left\{\left(\bar{\rho}_1 p_{x_1} - \frac{\dot{\bar{\rho}}_1}{\eta}x_1\right)^2 + \frac{\bar{C}_1}{\bar{\rho}_1^2}x_1^2\right\} + \left\{\left(\bar{\rho}_2 p_{y_1} + \frac{\dot{\bar{\rho}}_2}{\eta}y_1\right)^2 + \frac{\bar{C}_2}{\bar{\rho}_2^2}y_1^2\right\}\right] \tag{3.40}$$

where $C_i$ and $\bar{C}_i$, $i = 1, 2$ are arbitrary constants while $\bar{\sigma}_{1,2}(t)$ and $\bar{\rho}_{1,2}(t)$ are functions of the time satisfying the Pinney equation [17]

$$\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} - \frac{\dot{\eta}}{\eta}\frac{\mathrm{d}f}{\mathrm{d}t} + \eta^2 f = \frac{\eta^2\mathcal{B}}{f^3} \tag{3.41}$$

which has the solution

$$f(t) = \pm\left(\frac{1}{2}\mathcal{A} + \left[\frac{1}{4}\mathcal{A}^2 - \mathcal{B}\right]^{1/2} \sin\left(\mathcal{C} \pm 2\frac{\sqrt{1+\Gamma^2}}{\Gamma}\{\delta(t) - \delta(0)\}\right)\right)^{1/2} \tag{3.42}$$

where $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are constant parameters.

### 3.4. Complex invariants

In the previous work we have seen that to obtain real invariants we used several canonical transformations to reduce the Hamiltonian (3.2) to a simpler form. Now we seek for a complex invariant of the Hamiltonian (1.5) but without employing any transformations similar to those which we have used above. For this purpose we define the complex invariant $\hat{J}^{(2)}(t)$ in the form

$$\hat{J}^{(2)}(t) = \sum_{i=1}^{2}\left(\alpha_i \hat{a}_i^2 + \gamma_i \hat{a}_i^\dagger \hat{a}_i + \delta_1 \hat{a}_1 \hat{a}_2 + \delta_2 \hat{a}_1 \hat{a}_2^\dagger + \text{c.c.}\right) \tag{3.43}$$

where $\alpha_i(t), \delta_i(t), i = 1, 2$, are time-dependent complex functions while $\gamma_i(t), i = 1, 2$, are real functions depending upon time. Inserting equations (3.43) and (1.5) into the equation

$$\frac{\mathrm{d}\hat{J}^{(2)}(t)}{\mathrm{d}t} = \frac{\partial \hat{J}^{(2)}(t)}{\partial t} + \frac{1}{\mathrm{i}\hbar}[\hat{J}^{(2)}(t), H] = 0 \tag{3.44}$$

we obtain the following differential equations:

$$\frac{\mathrm{d}\alpha_1}{\mathrm{d}t} - 2\mathrm{i}\omega_1\alpha_1 = \mathrm{i}g\,\mathrm{e}^{\mathrm{i}\gamma}\delta_1 \qquad \frac{\mathrm{d}\alpha_2}{\mathrm{d}t} - 2\mathrm{i}\omega_2\alpha_2 = \mathrm{i}g\,\mathrm{e}^{-\mathrm{i}\gamma}\delta_1 \tag{3.45}$$

$$\frac{\mathrm{d}\delta_1}{\mathrm{d}t} - \mathrm{i}(\omega_1 + \omega_2)\delta_1 - 2\mathrm{i}g\,\mathrm{e}^{-\mathrm{i}\gamma}\alpha_1 - 2\mathrm{i}g\,\mathrm{e}^{\mathrm{i}\gamma}\alpha_2 = 0 \tag{3.46}$$

$$\frac{\mathrm{d}\delta_2}{\mathrm{d}t} + \mathrm{i}(\omega_2 - \omega_1)\delta_2 + \mathrm{i}g\,\mathrm{e}^{\mathrm{i}\gamma}(\gamma_1 - \gamma_2) = 0 \tag{3.47}$$

$$\frac{\mathrm{d}\gamma_1}{\mathrm{d}t} - \mathrm{i}g\,\mathrm{e}^{\mathrm{i}\gamma}\delta_2^* + \mathrm{i}g\,\mathrm{e}^{-\mathrm{i}\gamma}\delta_2 = 0 \qquad \frac{\mathrm{d}\gamma_2}{\mathrm{d}t} + \mathrm{i}g\,\mathrm{e}^{\mathrm{i}\gamma}\delta_2^* - \mathrm{i}g\,\mathrm{e}^{-\mathrm{i}\gamma}\delta_2 = 0. \tag{3.48}$$

Here we should note that in our procedure we have replaced $b$ and $c$ by $a_1$ and $a_2$ while $m\omega$ and $\omega_0$ have been replaced by $\omega_1$ and $\omega_2$, respectively. Thus, if we define $\zeta_1(t) = \sqrt{\alpha_1}\exp(-\mathrm{i}\omega_1 t)$, then we find that

$$\begin{aligned}\delta_1(t) &= -2\mathrm{i}\frac{\zeta_1(t)\dot{\zeta}_1(t)}{g(t)}\exp\left[\mathrm{i}(2\omega_1 t - \gamma(t))\right] \\ \alpha_2(t) &= -\frac{\dot{\zeta}_1^2(t)}{g^2(t)}\exp\left[2\mathrm{i}(\omega_1 t - \gamma(t))\right] + \frac{\tilde{C}_1}{\zeta_1^2(t)}\exp(2\mathrm{i}\omega_2 t)\end{aligned} \tag{3.49}$$

and hence the constant of the motion can be written as

$$\begin{aligned}\hat{J}^{(2)}(t) = &\left[\left(\zeta_1\hat{a}_1 - \mathrm{i}\frac{\dot{\zeta}_1}{g}\mathrm{e}^{-\mathrm{i}\gamma}\hat{a}_2\right)^2 \mathrm{e}^{2\mathrm{i}\omega_1 t} + \frac{\tilde{C}_1}{\zeta_1^2}\hat{a}_2^2\,\mathrm{e}^{2\mathrm{i}\omega_2 t}\right] \\ &+ \left[\left\{\left(\tilde{C}_3 \pm \sqrt{\tilde{C}_2 - |\delta_2|^2}\right)\hat{a}_1^\dagger\hat{a}_1 + \left(\tilde{C}_3 \mp \sqrt{\tilde{C}_2 - |\delta_2|^2}\right)\hat{a}_2^\dagger\hat{a}_2 + \delta_2\hat{a}_1\hat{a}_2^\dagger\right\} + \text{c.c.}\right]\end{aligned} \tag{3.50}$$

where the $\tilde{C}_i$, $i = 1, 2, 3$, are arbitrary constants.

Now we need to find the explicit expression of the functions $\alpha_i(t)$, $\delta_i(t)$ and $\gamma_i(t)$, $i = 1, 2$. This can be achieved if one takes $u(t)\exp\left[\mathrm{i}\int_0^t \Delta(t')\,\mathrm{d}t'\right]$ equal to the function $\zeta_1(t)$. In this case and after some calculation we find that the function $u(t)$ satisfies the equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} - \frac{\dot{g}}{g}\frac{\mathrm{d}u}{\mathrm{d}t} + \left[g^2 + \Delta^2 + \mathrm{i}\left(\dot{\Delta} - \frac{\dot{g}}{g}\Delta\right)\right]u = -\frac{g^2}{u^3}\tilde{C}_1. \tag{3.51}$$

A closed-form solution of equation (3.51) can be obtained provided we use the integrability condition $\Delta(t)/g(t) = \xi_1$. Reverting to $\alpha_1(t)$ we then have after some simple algebra the expression

$$\alpha_1(t) = \left[\frac{\xi_2}{2\left(1+\xi_1^2\right)} + \left[\frac{\tilde{C}_1}{1+\xi_1^2} + \frac{\xi_2^2/4}{\left(1+\xi_1^2\right)^2}\right]\sin\left[\xi_3 \pm 2\sqrt{1+\xi_1^2}k(t)\right]\right]$$
$$\times\exp\left(2\mathrm{i}\int_0^t \Delta(t')\,\mathrm{d}t' + \omega_1 t\right) \tag{3.52}$$

where $\xi_j$, $j = 1, 2, 3$, are constants. If one uses equation (3.49) together with equation (3.52), the explicit expression for the functions $\delta_1(t)$ and $\alpha_2(t)$ can be determined. On the other hand to obtain the functions $\gamma_i(t)$, $i = 1, 2$, and $\delta_2(t)$ we may use the fact that $\gamma_1(t)+\gamma_2(t) = \text{const}$. In this case the exact expression can be written as follows:

$$\gamma_{1,2}(t) = \zeta_0 \pm \frac{1}{\sqrt{1+\xi_1^2}}\sin\left[\zeta_1 \pm 2\sqrt{1+\xi_1^2}k(t)\right] \tag{3.53}$$

$$\delta_2(t) = \left[\zeta_2 + \mathrm{i}\zeta_3\exp\left[-\mathrm{i}\left(\zeta_1 + 2\sqrt{1+\xi_1^2}k(t)\right)\right]\right]\exp(\mathrm{i}\gamma(t)) \tag{3.54}$$

where $\zeta_j$, $j = 0, 1, 2, 3$, are arbitrary constants and $k(t)$ is the time-dependent function that has already been defined.

At this stage we turn our attention to consider another important point, that is the eigenfunctions and the corresponding eigenvalues for the constants of the motion. We present this in the following section.

## 4. Eigenfuctions and eigenvalues

In this section we attend to finding the eigenfunctions and the corresponding eigenvalues for the real quadratic invariants. To reach our goal we consider the quadratic invariant given by equation (3.39). It is noted that this invariant consists of two commutating parts and therefore, if we factorize each part into two complex product quantities and define the following operators:

$$\begin{aligned}\hat{F}_1 &= \left(2\hbar\sqrt{C_1/\eta(t)}\right)^{-1/2}\left\{\bar{\sigma}_1 x_1 + \left(\frac{\dot{\bar{\sigma}}_1}{\eta(t)} + \mathrm{i}\frac{\sqrt{C_1/\eta(t)}}{\bar{\sigma}_1}\right)p_{x_1}\right\}\\ \hat{F}_2 &= \left(2\hbar\sqrt{C_2/\eta(t)}\right)^{-1/2}\left\{\bar{\sigma}_2 y_1 - \left(\frac{\dot{\bar{\sigma}}_2}{\eta(t)} - \mathrm{i}\frac{\sqrt{C_2/\eta(t)}}{\bar{\sigma}_2}\right)p_{y_1}\right\}\end{aligned} \tag{4.1}$$

the constant of the motion in this case can be diagonalized and rewritten in the form

$$I^{(p)}(t) = \frac{2\hbar}{\sqrt{\eta(t)}}\left[\sqrt{C_1}\left(\hat{F}_1^\dagger\hat{F}_1 + \frac{1}{2}\right) + \sqrt{C_2}\left(\hat{F}_2^\dagger\hat{F}_2 + \frac{1}{2}\right)\right]. \tag{4.2}$$

This means that the operators (4.1) are the correct operators to diagonalize the constants of the motion. Therefore we employ them to find the eigenfunctions and the corresponding eigenvalues for the constants of the motion. It is interesting to point out here that the above Dirac operators, equation (4.1) together with their Hermitian conjugates may be regarded as annihilation and creation operators and satisfy the commutation relation $\left[\hat{F}_i, \hat{F}_j^{\dagger}\right] = \delta_{ij} = 1$ if $i = j$ and zero otherwise.

Now, if we define the coherent states

$$|\varsigma_1, \varsigma_2\rangle = \exp\left(-\frac{1}{2}[|\varsigma_1|^2 + |\varsigma_2|^2]\right) \sum_{n_1,n_2=0}^{\infty} \frac{\varsigma_1^{n_1}\varsigma_2^{n_2}}{\sqrt{n_1!n_2!}}|n_1, n_2\rangle \tag{4.3}$$

with the properties $\hat{F}_i|\varsigma_1, \varsigma_2\rangle = \varsigma_i|\varsigma_1, \varsigma_2\rangle$, then the expectation value for the invariant $I^{(p)}(t)$ equation (4.2) with respect to these states is

$$\langle I^{(p)}(t)\rangle = \frac{2\hbar}{\sqrt{\eta(t)}}\left[\sqrt{\mathcal{C}_1}\left(|\varsigma_1|^2 + \frac{1}{2}\right) + \sqrt{\mathcal{C}_2}\left(|\varsigma_2|^2 + \frac{1}{2}\right)\right]. \tag{4.4}$$

Before we proceed further we convert the operators (4.1) and express them in their physical coordinates and momenta. To do so we use the inverse transformation

$$\begin{bmatrix} x_1 \\ p_{x_1} \\ y_1 \\ p_{y_1} \end{bmatrix} = \begin{bmatrix} f_1 & f_3 & f_2 & -f_4 \\ -f_3 & f_1 & f_4 & f_2 \\ -g_2 & -g_3 & g_1 & -g_4 \\ g_3 & -g_2 & g_4 & g_1 \end{bmatrix} \begin{bmatrix} \sqrt{\omega_1}q_1 \\ p_1/\sqrt{\omega_1} \\ \sqrt{\omega_2}q_2 \\ p_2/\sqrt{\omega_2} \end{bmatrix} \tag{4.5}$$

where the $f_i, i = 1, 2, 3, 4$, are

$$\begin{aligned} f_1 &= \cos\delta_+ \cos\delta\cos\theta + \sin\delta_+ \sin\delta\sin\theta \\ f_2 &= \cos\delta_- \sin\delta\cos\theta - \sin\delta_- \cos\delta\sin\theta \\ f_3 &= \cos\delta_+ \sin\delta\sin\theta - \sin\delta_+ \cos\delta\cos\theta \\ f_4 &= \cos\delta_- \cos\delta\sin\theta + \sin\delta_- \sin\delta\cos\theta \end{aligned} \tag{4.6}$$

and the $g_i, i = 1, 2, 3, 4$, are

$$\begin{aligned} g_1 &= \cos\delta_- \cos\delta\cos\theta - \sin\delta_- \sin\delta\sin\theta \\ g_2 &= \cos\delta_+ \sin\delta\cos\theta + \sin\delta_+ \cos\delta\sin\theta \\ g_3 &= \cos\delta_+ \cos\delta\sin\theta - \sin\delta_+ \sin\delta\cos\theta \\ g_4 &= \cos\delta_- \sin\delta\sin\theta + \sin\delta_- \cos\delta\cos\theta. \end{aligned} \tag{4.7}$$

However, the expression of the operators in this case is complicated. Therefore to avoid any repetition we give the explicit expression for the wavefunction in terms of these physical quantities. From the above equations (4.1) and (4.5) and after simple algebra we have

$$\begin{aligned} \Psi_\varsigma(q_1, q_2, t) = N &\exp\left(\frac{\mathrm{i}\omega_1}{2\hbar l}\left[\{r_2(J_1 f_3 - \bar{\sigma}_1 f_1) + s_2(\bar{\sigma}_2 g_2 + J_2 g_3)\}q_1^2\right]\right) \\ &\times \exp\left(\frac{-\mathrm{i}\omega_2}{2\hbar l}\left[\{r_1(\bar{\sigma}_1 f_2 + J_1 f_4) + s_1(J_2 g_4 - \bar{\sigma}_2 g_1)\}q_2^2\right]\right) \\ &\times \exp\left(\frac{\mathrm{i}\sqrt{\omega_1\omega_2}}{\hbar l}[\{s_2(J_2 g_4 - \bar{\sigma}_1 g_1) - r_2(\bar{\sigma}_1 f_2 + J_1 f_4)\}q_1 q_2]\right) \\ &\times \exp\left(\frac{\mathrm{i}}{l}\sqrt{\frac{2\omega_1}{\hbar}}\left[\left\{\varsigma_1 r_2(\mathcal{C}_1/\eta)^{1/4} + \varsigma_2 s_2(\mathcal{C}_2/\eta)^{1/4}\right\}q_1\right]\right) \\ &\times \exp\left(\frac{\mathrm{i}}{l}\sqrt{\frac{2\omega_2}{\hbar}}\left[\left\{\varsigma_1 r_1(\mathcal{C}_1/\eta)^{1/4} - \varsigma_2 s_1(\mathcal{C}_2/\eta)^{1/4}\right\}q_2\right]\right) \end{aligned} \tag{4.8}$$

where $l = (s_1 r_2 + s_2 r_1)$ and $N$ is the normalization constant to be determined, while $s_i$, $r_i$ and $J_i(t)$, $i = 1, 2$, are given by

$$\begin{aligned} s_1 &= \bar{\sigma}_1 f_3 + J_1 f_1 & s_2 &= J_1 f_2 - \bar{\sigma}_1 f_4 \\ r_1 &= J_2 g_2 - \bar{\sigma}_2 g_3 & r_2 &= J_2 g_1 + \bar{\sigma}_2 g_4 \\ J_k(t) &= \frac{\dot{\bar{\sigma}}_k}{\eta(t)} \pm \mathrm{i}\frac{\sqrt{\mathcal{C}_k/\eta(t)}}{\bar{\sigma}_k} & k &= 1, 2. \end{aligned} \tag{4.9}$$

To complete our work we have to determine the value of the normalization constant $N$. This can be obtained from the relation

$$\int_{-\infty}^{\infty} |\Psi_\varsigma(q_1, q_2, t)|^2 \, \mathrm{d}q_1 \, \mathrm{d}q_2 = 1. \tag{4.10}$$

Therefore, if we insert equation (4.8) together with its complex conjugate into equation (4.10) and evaluate the integral, we attain our goal. However, it is noted that all the coefficients of the $q$ in this case are complex, and therefore to simplify matters we replace these quantities by $z_j = u_j + \mathrm{i}v_j$, $j = 1, \ldots, 5$. Thus after some calculation we have the following expression:

$$N = \frac{\left[\omega_1\omega_2\left(v_3^2 + v_1 v_2\right)\right]^{1/4}}{\sqrt{\pi\hbar}|l|} \exp\left[-\frac{\left(v_2 v_4^2 + 2v_3 v_4 v_5 - v_1 v_5^2\right)}{|l|^2\left(v_3^2 + v_1 v_2\right)}\right] \tag{4.11}$$

provided $v_1 > 0$, and $v_3^2 + v_1 v_2 < 0$, and this implies that the quantity $v_2$ must be negative in sign and its value greater than $v_3^2 / v_1$. The above normalization factor always holds as long as we guarantee that the constant $\mathcal{C}_k \neq 0$. This means that the coefficients of the $q$ are complex.

## 5. Conclusion

In the present paper we have introduced a Hamiltonian model which consists of a field–atom interaction. The system has been transformed to a time-dependent field–field interaction in a frequency converter form. We have handled the problem of constants of the motion for such a system in two different categories. Firstly we have concentrated on finding the real invariants and then secondly on obtaining the complex invariants. In each case we have considered both linear and quadratic invariants. Our main purpose in this work is to extend previous work (simple time-dependent harmonic oscillators) and to deal with two-dimensional real physical systems. In the meantime our purpose is also to direct the attention of other researchers to consider the invariant with its open classes of parameters instead of the Hamiltonian itself. This way is more complicated than the other to treat. However, advances in computational capacity may open the door for a deeper understanding of the physical problem.

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